Using Counting Techniques in Science

Connections

Have you ever . . .

- Tried to figure out how many kinds of sandwiches you can make using just what you have?
- Calculated the number of possible routes to a destination?
- Considered how many outfits you can create using the clothes you brought on vacation?

The counting principle helps you calculate the number of ways something can occur. If you know, for example, that you're going to be on vacation for two weeks, you can calculate that 7 tops and 2 pants will give you 14 different outfits.

The fundamental principle of counting is based on a few simple ideas. Which idea or rule applies depends on what you're counting and why.

- **The rule of multiplication:** The rule of multiplication states that if there are a ways for something to occur, and *b* ways for something else to occur, there are $a \times b$ ways for both things to occur together.
- **The rule of addition:** The rule of addition states that if there are a ways to do something, and also *b* ways to do something, but *a* and *b* cannot be done at the same time, the number of possible ways to do that thing is calculated with $a + b$.

When you use the counting principle, a **category** is a characteristic which can vary based on choice or circumstance, like color, weight, or height. A **variation** is a specific instance within a category. If the category is color, for example, a variation might be blue, red, or green.

A **combination** is a specific arrangement of variations. If you have two kinds of shirts and three pairs of pants, any pairing of a shirt and pants would be one combination. A **permutation** is a kind of combination where the variations of a category are sequenced or ordered.

Learn It!

Counting Variations

The counting principle can be used in science to determine variations within a group, such as a species. You can use simple math to understand variation.

Angelina is studying the Australian Orchidaceae, an orchid know for exhibiting variations in appearance. After looking at a large sampling of plants, Angelina notes three variations in size (small, medium, and large), four variations in petal arrangement (tight-and-narrow, spread-and-narrow, tight-and-wide, and spread-and-wide) and three variations in coloration (light, medium, and dark.)

How many possible combinations of Australian Orchidaceae can Angelina describe?

Determine the Categories

A category is something which can vary based on choice or circumstance. If you're making a sandwich, the categories might be your choice of bread, meat, and cheese. If you're looking at birds, your categories might be coloration, beak length, and wing span. If you're looking at cars, the categories you're interested in might be acceleration, braking, and gas mileage. There's no hard rule for what can or cannot be a category, except that it must be something that can vary. It all depends on what you're looking to count.

?

1. What are the categories that Angelina has identified in the Australian Orchidaceae?

Remember that a category is something whose qualities can vary. Usually a category is identified by variations that could represent this category. In this example, Angelina is using three categories: **size**, **petal arrangement**, and **coloration**. Notice that each of these categories can be represented with different variations. You might have a small, lightly-colored orchid, with tight-and-wide petals, or you might have a large, darkly colored orchid, with spread-and-wide petals.

For Each Category, Determine Variations

A category is defined by its potential to vary. This step is concerned with the number of variations within each category. If you were looking at dogs, for example, one of your categories might be size. You might decide, for your purposes, that a dog can be small (under 30 pounds), medium (30 to 60 pounds), or large (over 60 pounds). In this case, the number of variations for this category would be three: small, medium, or large.

? **2.** How many variations has Angelina identified for each category for the Australian Orchidaceae?

Angelina is using the categories size, petal arrangement, and coloration.

- For size, she has identified three variations: small, medium, and large.
- For petal arrangement, she has identified four variations: light-and-narrow, spread-and-narrow, tight-and-wide, and spread-and-wide.
- For coloration, she has identified three variations: light, medium, and dark.

Calculate the Possible Combinations

First, determine whether to use multiplication or addition. If all variations can occur together, multiply. If the variations cannot occur together, add.

? **3.** Using Angelina's data, what are the total number of combinations possible for the Australian Orchidaceae?

Angelina used three categories, and they respectively include 3, 4, and 3 possible variations. Each category can occur with the others, since color, petal arrangement, and size are independent of each other. Multiply these the variations together to find the possible combinations. There are 36 combinations:

$$3 \times 4 \times 3 = 36$$

Practice It!

Use counting techniques to answer the following questions.

1. Quentin is teaching himself how to create scientific illustrations of dogs. Currently he's focused on sketching the head. Quentin has noted that dogs generally either have wedge-shaped heads or conical-shaped heads. They also either have a long or short muzzle-shape. Finally, Quentin has also noted that dogs have ear-shapes that include pricked-up, folded, or floppy.

a. What categories can Quentin use to describe dogs?

b. How many variations are in each category?

c. What is the total number of combinations for the characteristics of a dog's head?

2. Cassandra visits a local cove to observe the behavior of sea birds. She notes the behavior of several species: pelicans, gulls, coots, ducks, and geese. During her visit she sees birds moving through different terrain. Some birds walk down the shoreline. Other birds prefer to swim in the ocean. Many of the birds seem to determined to get some rest and relaxation. Cassandra photographs sea birds sunning themselves in rocky outcroppings or sleeping in tall grass.

a. What categories can Cassandra use to describe sea birds?

b. How many variations are in each category?

c. What is the total number of combinations for the characteristics of sea birds and their terrain?

3. Nikolai's cat, Sushi, is expecting kittens. Based on the kitten's parents and grandparents, Nikolai has some idea what the kittens might look like. He expects their eyes will be yellow, green, or blue. Although most of the lineage has short hair, Sushi's mother had long hair, so it's possible that one or more of the kittens might inherit this characteristic. Based on the kittens' pedigree, they could have a wide variety of ear-shapes: lynx-tipped, oval, folded, or triangular. The kittens could have tails that are long and slender, like their father's side of the family, or short and fluffy like Sushi.

a. What categories can Nikola use to describe the kittens?

b. How many variations are in each category?

c. What is the total number of combinations for the characteristics of the kittens?

4. A meadow has blue and yellow flowers and tall and short grasses.

a. What categories can you use to describe the plants in the meadow?

b. How many variations are in each category?

c. What is the total number of combinations for plants in the meadow? Explain your reasoning for how you calculated the total number.

Learn It!

Calculating Permutations

Permutations are combinations where order is importabt. If you know the digits of a phone number, the correct number is a permutation: the numbers in order.

Arturo knows that a small sequence of DNA contains two A nucleotides, two G nucleotides, one C nucleotide, and one T nucleotide.

How many possible permutations are there of the DNA sequence?

Define the Number of Items in the Group

Think of a permutation as a group of people standing in line. The number of positions in the line is the size of the group. How many total items are there in the group?

?

1. How many items are there in the total DNA sequence?

The DNA sequence has six total nucleotides. Even though there are four types of nucleotides, there are six total nucleotides in the whole group.

Multiply Down the Line

In a permuation without repetition, you need to fill each space in the "line." When you fill the first "space," you have every member of the group to choose from. That's your starting number: the total number of items in the group.

When you fill the next space, though, one of the items in the group is already in line. You have the total number of items minus one to choose from. For each space in line, you have one less item to choose from, until you're out of items.

To find the total permutations, according to the counting principle you need to multiply the number of options for each space in the "line." So, for a group of x items, you'll multiply all the numbers from x down to 1, subtracting 1 each time.

$$x \times (x-1) \times (x-2) \ldots \times 2 \times 1$$

?

2. Ignoring that two of the nucleotides are the same, how many permutations are there for the sequence?

The sequence has six nucleotides, so multiply every number from six down to one.

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

In math, this type of expression is indicated with an exclamation point: $6! = 720$

Each number you're multiplying is one item in the line: to fill the first spot, you have six nucleotides to choose from. To fill the second spot, you have five nucleotides to choose from. When you get to the last spot, you only have one nucleotide left to fill the spot.

Consider Conditions on the Problem

The problem might be a simple permutation, or the problem might have other conditions. Consider the conditions. How do they change the answer?

In this problem, because two of the nucleotides are interchangeable, some of the 720 permutations will be the same. How many? Consider the letters MOM. How many unique arrangements are there? The O will be in the first, second, or third position, and the two Ms will be in the other positions. There are six total possibilities and three unique possibilities:

OMM OMM MOM MOM MMO MMO

It turns out that this amount is (letters)! divided by (repeated letters)! , or $3! \div 2! = 6 \div 2 = 3$. If there are two sets of repeated letters, multiply (repeated letter a)! by (repeated letter b)! to get the number to divide by.

3. Because two sets of two nucleotides are interchangeable, some of the 720 possible permutations will be the same. How many unique permutations are there?

If $n =$ total spots in the sequence, $\mathrm{a} =$ repetitions of nucleotide A, and $\mathrm{g} =$ repetition of nucleotide G, to find the total unique permutations, the calculation is:

$$\frac{n!}{a!g!}$$

Since there are six total spots in the sequence and two repetitions of nucleotides A and G, there are 180 total unique combinations.

$$\frac{6!}{2!2!} = \frac{720}{2 \times 2} = \frac{720}{4} = 180$$

Practice It!

Use permutations to answer the following questions.

1. Alex will put eight plants in a row under high-output lights as part of an experiment measuring the effects of light on growth.

a. How many possible arangements of plants are there in Alex's experiment? Explain your calculations.

b. Alex takes four plants away from the lights for twenty-four hours. If he puts the four plants back in no particular order, how many possible re-arrangements of the plants are there?

2. Joanne knows that a sequence of four nucleotides contains one T nucleotide, one G nucleotide, one A nucleotide, and one C nucleotide.

a. How many possible arrangements of nucleotides are there in the sequence?

b. Joanne discovers that the second spot in the sequence is the T nucleotide. Knowing this, how many possible arrangements are there? Explain how you arrived at your answer.

2. Grace is conducting an experiment to see where strangers will sit in relation to each other in a room. She has four volunteers all arrive in a room with eight chairs.

a. How many possible arrangements are there of specific volunteers on the chairs? Four chairs will be full, and four will be empty. Explain your calculations.

b. How many possible arrangments are there if there are five volunteers? How does this compare to four volunteers?

c. If there were eight volunteers but four chairs, only four volunteers could sit. Where p is the total number of people and c is the total number of chairs, the formula for this is:

$$\frac{p!}{(p-c)!}$$

How many possible arrangments of volunteers are there on the four chairs?

d. Explain why this answer is correct, or calculate the answer in a different way and explain your reasoning.

4. Carmine has three yellow pansy plants, two blue ones, and two purple ones. She is placing them in a row under high-output lights as part of an experiment.

   a. How many permutations are there for the plants if Carmine can use any order?

   b. How many permutations are there for the plants if Carmine wants to put the yellow plants on the left, the blue ones in the center, and the purple ones on the right? Explain your reasoning.

   c. How many permutations are there for the plants if Carmine wants to keep all plants of the same color together, but in any order of colors? Explain your reasoning.

Build Your Math Skills

If a permutation allows repetitions, like a 3-letter string, just multiply the number of possibilities for each position. With three positions and 26 possible letters:

$26 \times 26 \times 26 = 17{,}576$

Check Your Skills

When you see this icon, you may use a calculator.

Read the problem and select the choice that best answers the question. Remember to approach each problem using the steps for unpacking and solving formulas.

1. A preserve has eight animal species living on a plot of land. Each night for three days, Tom captures a photo of one of the species animals on a night camera he has set up to record animal activity in the area. How many different combinations of animal species might Tom have photographed?

   a. 1,536

   b. 512

   c. 336

   d. 24

2. Tracy finds that a small pond contains minnows in three size ranges, under 5 cm, 5 to 10 cm, and 10 to 15 cm. Some are all silver, while others are silver with green and blue markings. Size does not seem related to color. How many types of minnows with combinations of these different traits are there?

   a. 3

   b. 4

   c. 5

   d. 6

3. A scientists takes samples from two plots of soil and measure the nitrogen content of the samples. The samples are kept in a row on a shelf.

Site	Sample No.	Nitrogen
A	1	5.2%
A	2	5.5%
B	1	4.2%
B	2	4.4%
B	3	4.0%

   If the A samples are kept on the right and B samples are on the left, how many different possible permutations of samples are there on the shelf?

   a. 8

   b. 12

   c. 30

   d. 120

4. A beach contains a combination of white and yellow rocks and white and gray shells, either broken or intact. How many possible types of rocks and shells are available on the beach?

   a. 4
   b. 6
   c. 8
   d. 12

5. Six hermit crabs are put in a cage with six shells, and each hermit crab takes one shell. How many possible sets of specific hermit crabs in speciifc shells are there?

   a. 36
   b. 80
   c. 120
   d. 720

6. A researcher has a system of identification numbers for samples that begins with two upper-case letters, followed by three numbers (0 to 9). How many possible samples can the researcher label this way?

   a. 1,300
   b. 1,676
   c. 676,000
   d. 829,000

7. A researcher gives a true-false personality profile to volunteers. The questionaire has 10 questions. If each question is answered, how many possible answer combinations are there?

   a. 20
   b. 512
   c. 1,024
   d. 2,048

Remember the Concept

- Multiply to find the number of conditions that can occur together.
- Add if conditions can't occur together.
- "Multiply down the line" to find permutations: $x!$

Using Counting Techniques in Science

Counting Variations

Practice It!

1a. There are three categories: head shape, muzzle shape, and ear shape.

1b. For head shape, there are two variations. For muzzle shape, there are two variations. For ear shape, there are three variations.

1c. 12 combinations

$2 \times 2 \times 3 = 12$

2a. There are two categories: species and terrain.

2b. For species, there are five variations: pelicans, gulls, coots, ducks, and geese.

For terrain, there are four variations: shoreline, ocean, rocky outcroppings, and tall grass.

2c. 20 combinations

$5 \times 4 = 20$

3a. There are four categories: eye color, fur length, ear shape, and tail shape.

3b. For eye color, there are three variations. For fur length, there are two variations. For ear shape, there are four variations. For tail shape, there are two variations.

3c. 48 combinations

$3 \times 2 \times 4 \times 2 = 48$

4a. There are two categories: color and height.

4b. Each category has two variations. For color, the variations are blue and yellow. For height, the variations are tall and short.

4c. 4 combinations

Because only flowers are blue and yellow, and only grasses are tall and short, the variations shouldn't be multiplied. They don't appear together. Since they appear separately, they should be added.

You can list the combinations: blue flowers, yellow flowers, tall grass, and short grass.

$2 + 2 = 4$

Calculating Permutations

Practice It!

1a. 40,320 permuations

There are eight plants in a row, so for the first spot there are eight possibilities. For the second spot, there are seven possibilities. For the third spot their are six possibilities, and so one down to one possibility in the final spot. Multiply the possibilities together to find the total number of permutations.

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

1b. 24 permutations

When Alex takes away four plants and puts them back, only four plants are rearranged in four places. The other plants are static, in the same place. The calculation is a permutation of four items.

$4! = 4 \times 3 \times 2 \times 1 = 24$

Notice how big a difference there is between a permutation of four objects and a permutation of eight objects.

2a. 24 permutations

The sequence is a permutation of four unique nucleotides in a series.

$4! = 4 \times 3 \times 2 \times 1 = 24$

2b. 6 permutations

If Joanne knows which nucleotide is in one spot, that position is static. It doesn't change. The remaining possible permutations are calculated for the remaining three nucleotides.

$3! = 3 \times 2 \times 1 = 6$

3a. 1,680 permutations

There are eight chairs, so the total permutations of eight unique items would be:

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

However, four chairs are empty, so they are the same. Divide by $4!$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$8! \div 4! = 40{,}320 \div 24 = 1{,}680$

3b. 1,680 permutations

There are still eight chairs, so the total permutations of eight unique items would be:

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

However, in this case three chairs are empty. Divide by $3!$

$3! = 3 \times 2 \times 1 = 6$

$8! \div 3! = 40{,}320 \div 6 = 6{,}720$

The total number of permutations is significantly higher because there are fewer non-unique elements (empty chairs). The more non-unique elements there are, the fewer total possible permutations there will be.

3c. 1,680 permutations

There are eight people, so the total permutations of eight unique people would be:

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40{,}320$

Divide by $(people - chairs)!$. Divide by $4!$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$8! \div 4! = 40{,}320 \div 24 = 1{,}680$

3d. You can think through how to find the number of permutations of people in chairs. For the first chair, there's a possible eight people to sit in it. For the second chair, there's a possible seven people, then six people, then five people. Since there's only four chairs, stop there.

$8 \times 7 \times 6 \times 5 = 1{,}680$

You take away the four numbers at the end of the calculation, because you're missing four chairs. This is the same as $8! \div 4!$, as you can see with cancallation.

$$\frac{8 \times 7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 8 \times 7 \times 6 \times 5$$

4a. 5,040 permutations

There are seven total plants, so the total permutations would be:

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5{,}040$

4b. 24 permutations

If the sets of plants are kept together, then you need to calculate the total permutations of each set of plants first. This gives you the total number of possible orderings of plants in each group.

Three yellow pansy plants $= 3! = 3 \times 2 \times 1 = 6$

Two blue pansy plants $= 2! = 2 \times 1 = 2$

Two purple pansy plants $= 2! = 2 \times 1 = 2$

You still need to account for combinations of these permutations on the shelf. It's not a calculation of permutation, because the order of yellow-blue-purple is set. Using the counting principle of multiplication, multiply the results to get the total number of permutations.

$6 \times 2 \times 2 = 24$

4c. 144 permutations

If the sets of plants can be in any order, then there is an additional permutation. You need to calculate the total permutations of each set of plants, but you also need to calculate the permutation of orders of colors.

Three yellow pansy plants $= 3! = 3 \times 2 \times 1 = 6$

Two blue pansy plants $= 2! = 2 \times 1 = 2$

Two purple pansy plants $= 2! = 2 \times 1 = 2$

Orders of three colors $= 3! = 3 \times 2 \times 1 = 6$

Using the counting principle of multiplication, multiply all of these categories together to get the total number of possible permutations.

$6 \times 2 \times 2 \times 6 = 144$

Check Your Skills

1. b. 512

Any of the eight species can be photographed on any night. There are no restrictions. Each night, one of eight species can be photographed. Using the counting principle, multiply eight by itself three times.

$8^3 = 8 \times 8 \times 8 = 512$

2. d. 6

There are two categories that describe the minnows: size and color.

There are three variations of size: under 5 cm, 5 to 10 cm, and 10 to 15 cm.

There are two variations of color: silver and silver with blue and green.

Multiply the number of variations of each category to find the number of combinations of traits:

$3 \times 2 = 6$

3. b. 12

There are two groups of samples, samples from site A and samples from site B. First, find the number of permutations within each sample group

$A = 2! = 2 \times 1 = 2$

$B = 3! = 3 \times 2 \times 1 = 6$

The groups are in a set order, so you don't need to calculate a permutation of the number of groups. Using the counting principle of multiplication, multiply the number of permutations for each sample set to get the total number of permutations.

$6 \times 2 = 12$

4. b. 6

Consider rocks and shells separately, because there can't be something that's both a rock and a shell.

There is one category for rocks (color), with two variations, so there are two possible types of rocks.

There are two categories for shells. Shells have two color variations, and two shape variations (broken and unbroken). There are four possible categorizations of shells.

Since shells are rocks are separate, use the addition principle to find the total number of types of objects.

$2 + 4 = 6$

5. d. 720

This is a permutation problem. You're trying to find a specific order of crabs in shells. When the first crab takes a shell, one out of six crabs might take that shell. When the second crab takes a shell, there are only seven crabs left. When the last crab takes a shell, there's only one crab. Since there are six crabs and six possible positions, there are 720 permutations:

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

6. c. 676,000

For the letters, since there are 26 letters in the alphabet, there are 26 possible letters in each position. Since the letters can be repeated, and there's not one less letter to choose from for each position, the calculation is simply:

letters $= 26 \times 26 = 676$

For the numbers, there are 10 possible digits (0 thorugh 9) in each position. Since the numbers can be repeated, and there's not one less digit to choose from for each of three position, the calculation is:

numbers $= 10 \times 10 \times 10 = 1{,}000$

This makes sense, because the possible numbers will be from a thousand possibilites: from 000 to 999.

For each set of letters, there can be one corrresponding set of numbers. Using the counting principle of multiplication, multiply the number of possibilities of letters and numbers together:

$676 \times 1{,}000 = 676{,}000$

7. a. 1,024

The number of possible answers in each position is two. The number of positions is ten. The number of possible answers doesn't change for each position, so simply multiply the number of possible answers by itself ten times.

$2^{10} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 1{,}024$